CS103 Review Session

Studying for the Final

General strategies

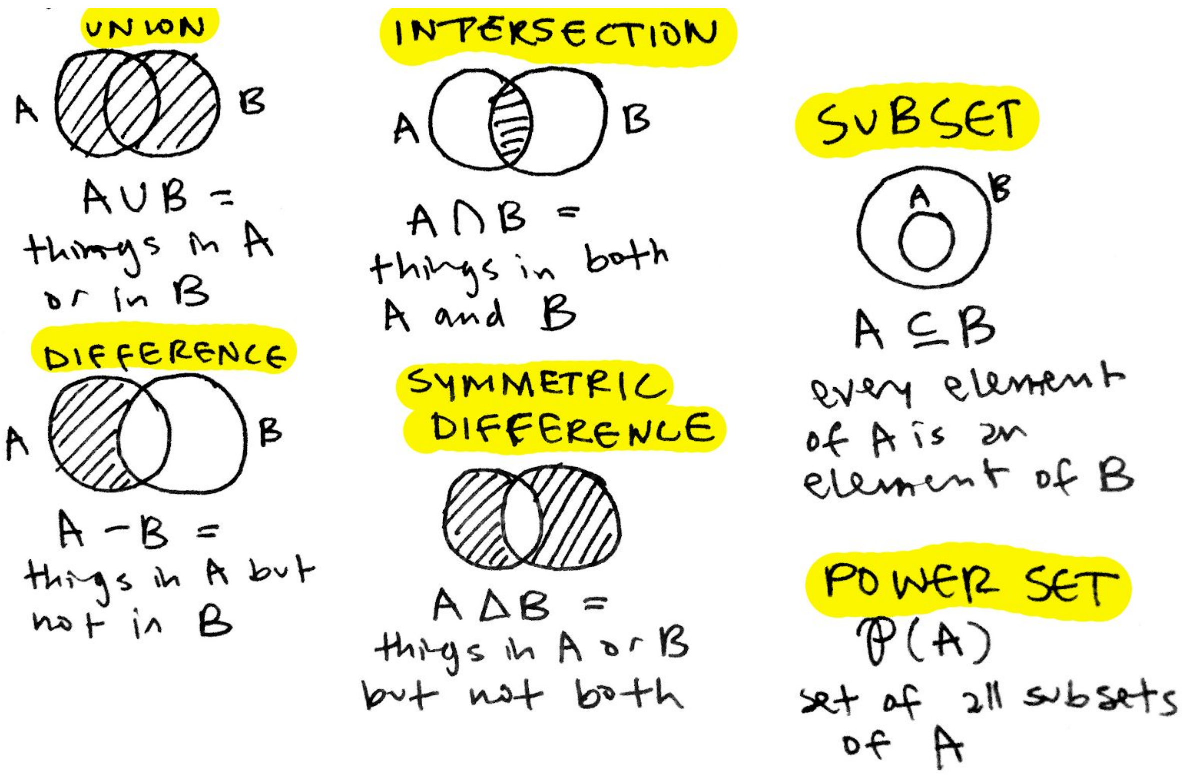
- Write out everything you know and what you’re trying to prove.
- What is the quantifier on the statement you’re trying to prove? What does that tell you about how the proof should be set up?
- What kind of structure are you trying to reason about? (binary relations, sets, functions, etc.) You know how to write proofs about all of these! Use proof templates and formal definitions to guide you.
- What proof strategy are you using? What do you get to assume? If you’re doing an indirect proof, would it be helpful to write out the statement in FOL and negate it?

General strategies

- Make sure you're using all parts of what's given to you! Usually there's a good reason why you need each assumption/condition to get the proof to work.
- Draw pictures! Work backwards! Try a different proof strategy! It's okay if the first thing you try doesn't work, just try something else!

Set Theory

- Union
- Intersection
- Difference
- Symmetric difference
- Subset
- Power set



Proofs about sets

To show $A \subseteq B$:

- Pick an arbitrary $x \in A$
- Show that $x \in B$

To show $A = B$:

- Prove $A \subseteq B$
- Prove $B \subseteq A$

Example set theory proof

Let A and B be arbitrary sets. Prove that $A \in p(B)$ if and only if $A \cap B = A$.

Example set theory proof

Let A and B be arbitrary sets. Prove that $A \in p(B)$ if and only if $A \cap B = A$.

What is A?

What is B?

to show $A = B$:

- prove $A \subseteq B$.
- prove $B \subseteq A$.

Proof 1: We will prove both directions of implication. First, we'll prove that if $A \in \wp(B)$, then $A \cap B = A$. To do so, we'll prove both $A \cap B \subseteq A$ and $A \subseteq A \cap B$.

Let's begin by showing that $A \cap B \subseteq A$. To do so, pick any $x \in A \cap B$. This means in that $x \in A$, and since our choice of x was arbitrary, we conclude that $A \cap B \subseteq A$, as needed.

Next, we'll show that $A \subseteq A \cap B$. Consider any $x \in A$. We will prove that $x \in A \cap B$. We know $A \in \wp(B)$, which means that $A \subseteq B$. Since $x \in A$ and $A \subseteq B$, we see that $x \in B$. Then, since $x \in A$ and $x \in B$, we see that $x \in A \cap B$, as required.

For the other direction of implication, assume that $A \cap B = A$. We will prove that $A \in \wp(B)$. To do so, we will prove that $A \subseteq B$. So pick any $x \in A$. Then since $x \in A$ and $A = A \cap B$, we see that $x \in A \cap B$. Therefore, we see that $x \in B$. Since our choice of $x \in A$ was arbitrary, we see that $A \subseteq B$, as required. ■

Types of proofs

- Universal statements: “for all x…”
  - Proof: Pick an arbitrary x, and show that the statement is true
  - Disproof: find a counter-example
- Existential statements: “there is an x such that…”
  - Proof: Find an example
  - Disproof: Pick an arbitrary x and show that the statement is false
- Implications “$P \to Q$”
  - **Directly:** assume P and prove Q
  - **By contrapositive** ($!Q \to !P$): assume !Q and prove !P
- Proof by contradiction:
  - Assume !P, arrive at a contradiction

First Order Logic

- Can help unpack or take the negation of statements we are trying to prove
- “All P’s are Q’s”: $\forall x.\ P(x) \to Q(x)$
- “No P’s are Q’s”: $\forall x.\ P(x) \to \neg Q(x)$
- “Some P’s are Q’s”: $\exists x.\ P(x) \land Q(x)$
- “Some P’s are not Q’s”: $\exists x.\ P(x) \land \neg Q(x)$
- “For any choice of x, there is some y such that P(x,y) is true”: $\forall x \exists y.\ P(x,y)$
- “There is some x where for any choice of y, P(x,y) is true”: $\exists x \forall y.\ P(x,y)$

First Order Logic

- $\forall$ is usually paired with $\rightarrow$
- $\exists$ is usually paired with $\land$
- Existential statements are false unless there is a positive example
- Universal statements are true unless there is a counter example

Binary Relations

- Reflexive
  - $\forall a \in A.\ a\,R\,a$
- Symmetric
  - $\forall a \in A.\ \forall b \in A.\ aRb \to bRa$
- Transitive
  - $\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ aRb \land bRc \to aRc$
- Irreflexive
  - $\forall a \in A.\ a \not R\, a$
- Asymmetric
  - $\forall a \in A.\ \forall b \in A.\ aRb \not\to bRa$

Binary Relations

- Reflexive
  - $\forall a \in A.\ a\ R\ a$
  - <u>Proof setup</u>: pick an $a \in A$. Show aRa.
- Symmetric
  - $\forall a \in A.\ \forall b \in A.\ aRb \rightarrow bRa$
  - <u>Proof setup</u>: pick an $a \in A$ and $b \in A$ such that aRb. Show bRa.
- Transitive
  - $\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ aRb \land bRc \rightarrow aRc$
  - <u>Proof setup</u>: pick an $a,b,c \in A$ such that $aRb \land bRc$. Show that aRc.

Binary Relations

- Irreflexive
  - $\forall a \in A.\ \cancel{a R} a$
  - Proof setup: pick an $a \in A$. Show $\cancel{aR}a$.
- Asymmetric
  - $\forall a \in A.\ \forall b \in A.\ aRb \to \cancel{b}Ra$
  - Proof setup: pick an $a \in A$ and $b \in A$ such that aRb. Show $\cancel{b}Ra$.

Binary Relations

- Equivalence Relations
  - Reflexive, symmetric, and transitive
- Strict Orders
  - Irreflexive, asymmetric, and transitive
  - OR equivalently, irreflexive and transitive
  - OR equivalently, asymmetric and transitive

Example binary relations proof

If R_1 is a binary relation over a set A_1 and R_2 is a binary relation over a set A_2, then an ***embedding of R_1 in R_2*** is a function $f : A_1 \to A_2$ such that

$$\forall a \in A_1.\ \forall b \in A_1.\ (aR_1b \leftrightarrow f(a)\ R_2\ f(b)).$$

If there's an embedding of a relation R_1 in a relation R_2, we say that R_1 ***can be embedded in*** R_2.

> Let R_1 be a binary relation over a set A_1 and let R_2 be a *strict order* over some set A_2. Prove that if R_1 can be embedded in R_2, then R_1 is a strict order.

IRREFLEXIVE

$\forall a \in A.\ a \not R a.$

no element is related to itself

Proof setup:

Pick an $a \in A$. Prove $a \not R a$.

What set is the relation defined over?
(Where should we be picking our arbitrary element from?)

Example binary relations proof

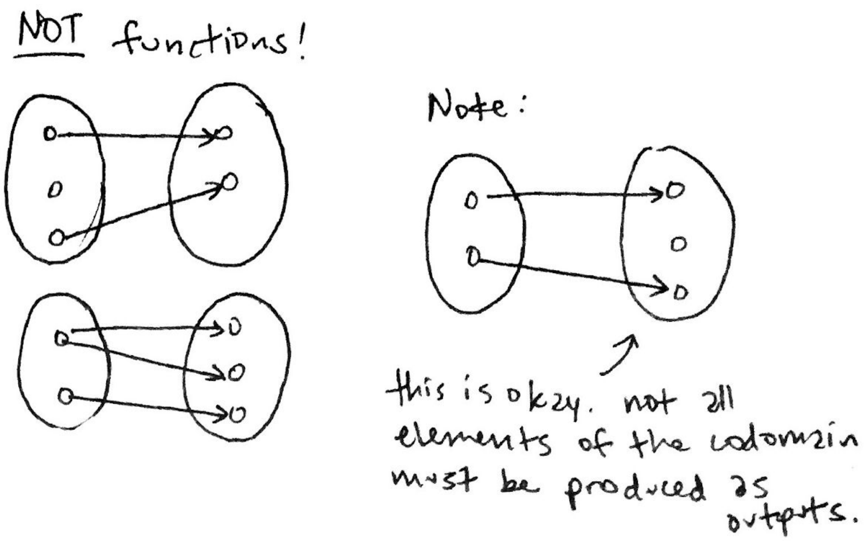
Proof 1: Let $f : A_1 \to A_2$ be an embedding of R_1 in R_2. We will show that R_1 is a strict order by proving that it is irreflexive and transitive.

First, we'll show that R_1 is irreflexive. Consider any $a \in A_1$. Since R_2 is a strict order, we know that R_2 is irreflexive, so $f(a) \not R_2 f(a)$. Then, since f is an embedding of R_1 in R_2, we see that $a \not R_1 a$, as required.

Next, we'll show that R_1 is transitive. To do so, consider any $a, b, c \in A_1$ where aR_1b and bR_1c. Since f is an embedding of R_1 in R_2, we then see that $f(a)R_2 f(b)$ and $f(b)R_2 f(c)$. Then, since R_2 is a strict order, it's transitive, and so $f(a)R_2 f(c)$. Finally, since f is an embedding of R_1 in R_2, we use the reverse direction of the implication to conclude that aR_1c, as required. ■

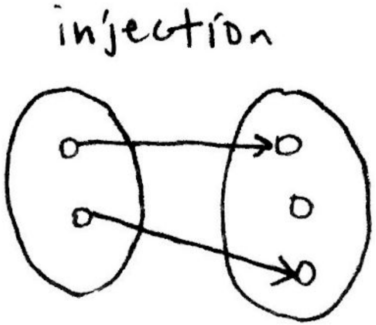
Functions

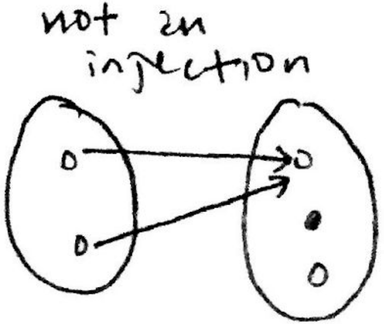
- All functions: f: A→B
  - Every input maps to some output
    - For all *a* in A, there exists *b* in B such that f(a) = b.
  - Functions are deterministic: equal inputs produce equal outputs
    - For all a1, a2 in A if a1 = a2, then f(a1) = f(a2).



Functions

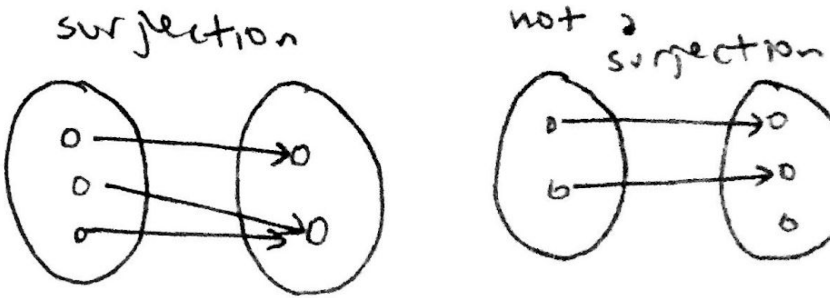
- Injective functions
  - Different inputs produce different outputs
  - For all a1, a2 in A, $a1 \neq a2 \rightarrow f(a1) \neq f(a2)$.
  - For all a1, a2 in A, $f(a1) = f(a2) \rightarrow a1 = a2$.





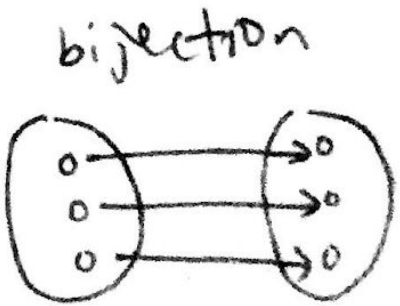
Functions

- Surjective functions
  - For every possible output, there exists at least one possible input that produces it.
  - For all *b* in B, there exists an *a* in A such that *f(a)* = *b*.



Functions

- Bijective functions
  - Functions that are both injective and surjective



Example of function proof

Imagine you have a function $f : A \to B$ from some set A to some set B. We can use f to construct a new function called the ***lift of f***, denoted $\textbf{lift}_f$, from $\wp(A)$ to $\wp(B)$. Specifically $\text{lift}_f : \wp(A) \to \wp(B)$ is defined as follows:

$$\text{lift}_f(S) = \{\ y \mid \exists x \in S.\ f(x) = y\ \}$$

Let A and B be sets. Prove that if $f : A \to B$ is injective, then lift_f is injective.

Injective functions $f : A \to B$

- $\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$
  different inputs produce different outputs

What function are we trying to prove things about?

What is the domain of that function?

Example of function proof

Proof 1: Let $f : A \to B$ be an injective function. We will prove that lift_f is injective as well. To do so, consider any $S_1, S_2 \in \wp(A)$ where $S_1 \neq S_2$. We will prove that $\text{lift}_f(S_1) \neq \text{lift}_f(S_2)$.

Since $S_1 \neq S_2$, we know that either $S_1 \not\subseteq S_2$ or that $S_2 \not\subseteq S_1$. Without loss of generality, assume $S_1 \not\subseteq S_2$, which means that there is some $a \in S_1$ where $a \notin S_2$.

First, notice that since $a \in S_1$, we see that $f(a) \in \text{lift}_f(S_1)$. We now claim that $f(a) \notin \text{lift}_f(S_2)$. To see this, suppose for the sake of contradiction that $f(a) \in S_2$. This means that there must be some $a' \in S_2$ such that $f(a') = f(a)$. Since f is injective, that tells us that $a' = a$, and since $a' \in S_2$, we see that $a \in S_2$ as well. But this is impossible, since we know that $a \notin S_2$. We've reached a contradiction, so our assumption was wrong and $f(a) \notin \text{lift}_f(S_2)$.

Since $f(a) \in \text{lift}_f(S_1)$ but $f(a) \notin \text{lift}_f(S_2)$, we see $\text{lift}_f(S_1) \neq \text{lift}_f(S_2)$, which is what we needed to show. ■

5 minute break!

The Pigeonhole Principle

Pigeonhole Principle Refresher

- The ***generalized pigeonhole principle*** says that if you distribute m objects into n bins, then
  - some bin will have at least $\lceil {}^{m}/_{n} \rceil$ objects in it, and
  - some bin will have at most $\lfloor {}^{m}/_{n} \rfloor$ objects in it.

$\lceil {}^{m}/_{n} \rceil$ means "${}^{m}/_{n}$, rounded up."
$\lfloor {}^{m}/_{n} \rfloor$ means "${}^{m}/_{n}$, rounded down."

Pigeonhole Principle Clues + Tips

1) Look for **"at most"**, **"at least", "less than", or "more than"** in the problem statement. It's not a guarantee but often pigeonhole principle problems use these terms.

Let's begin with some new definitions. First, we'll say that a ***matching*** in a graph $G = (V, E)$ is a set $M \subseteq E$ of edges in G such that no two edges in M share an endpoint. The ***size*** of a matching is the number of edges it contains. The ***matching number*** of a graph G, denoted $\upsilon(G)$, is the size of the largest matching in G.

Now, let's introduce a variation on a definition we've seen before. A ***k-edge coloring*** of a graph $G = (V, E)$ is a way of coloring each of the edges in G one of k different colors so that no two edges that share an endpoint are assigned the same color. The ***chromatic index*** of a graph G, denoted $\chi_1(G)$, is the minimum number of colors needed in any edge coloring of G.

Let G be an undirected graph with exactly n^2+1 edges for some natural number $n \geq 1$. Prove that either $\chi_1(G) \geq n+1$ or $\upsilon(G) \geq n+1$ (or both).

Pigeonhole Principle Clues + Tips

2) Try writing out all the nouns mentioned in the problem statement and their quantity (if known).

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graph (1)
vertices (?)
edges (n^2 +1)
edges in largest matching (at most n^2 +1)
colors (at most n^2 +1)

Most of the time:

There are more pigeons than holes
There is more than one hole
We know how many holes and pigeons there are
(otherwise the result isn't very interesting...)

Pigeonhole Principle Clues + Tips

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graph (1)
vertices (?)
edges (n^2 +1) **maybe the pigeons?!** **maybe the holes?!**
edges in largest matching (at most n^2 +1)
colors (at most n^2 +1) **maybe the holes?!**

Most of the time:

There are more pigeons than holes
There is more than one hole
We know how many holes and pigeons there are
(otherwise the result isn't very interesting...)

Example Pigeonhole Principle Proof

Let's begin with some new definitions. First, we'll say that a ***matching*** in a graph $G = (V, E)$ is a set $M \subseteq E$ of edges in G such that no two edges in M share an endpoint. The ***size*** of a matching is the number of edges it contains. The ***matching number*** of a graph G, denoted $\upsilon(G)$, is the size of the largest matching in G.

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Let G be an undirected graph with exactly n^2+1 *edges* for some natural number $n \geq 1$. Prove that either $\chi_1(G) \geq n+1$ or $\upsilon(G) \geq n+1$ (or both).

How do you prove a statement of the form P or Q?

Example Pigeonhole Principle Proof

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Let G be an undirected graph with exactly n^2+1 *edges* for some natural number $n \geq 1$. Prove that either $\chi_1(G) \geq n+1$ or $\upsilon(G) \geq n+1$ (or both).

Proof: Let G be an arbitrary undirected graph with n^2+1 edges for some positive natural number n. We will prove that if $\chi_1(G) \leq n$, then $\upsilon(G) \geq n+1$.

Suppose that $\chi_1(G) \leq n$. This means that there is an n-edge coloring of the graph G. Since there are n^2+1 edges and n colors, by the generalized pigeonhole principle we know that there must be at least $\lceil (n^2+1) / n \rceil = \lceil n + {}^1/_n \rceil = n+1$ edges that are all the same color in the n-edge coloring. Since all those edges are assigned the same color, we know that no two of them can share an endpoint. Therefore, this set of $n+1$ edges forms a matching, so $\upsilon(G) \geq n+1$, as required. ■

Induction

Induction Refresher

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true…

$\boldsymbol{P(0)}$ **is true**

…and it stays true…

and

$$\forall k \in \mathbb{N}.\ (P(k) \rightarrow P(k+1))$$

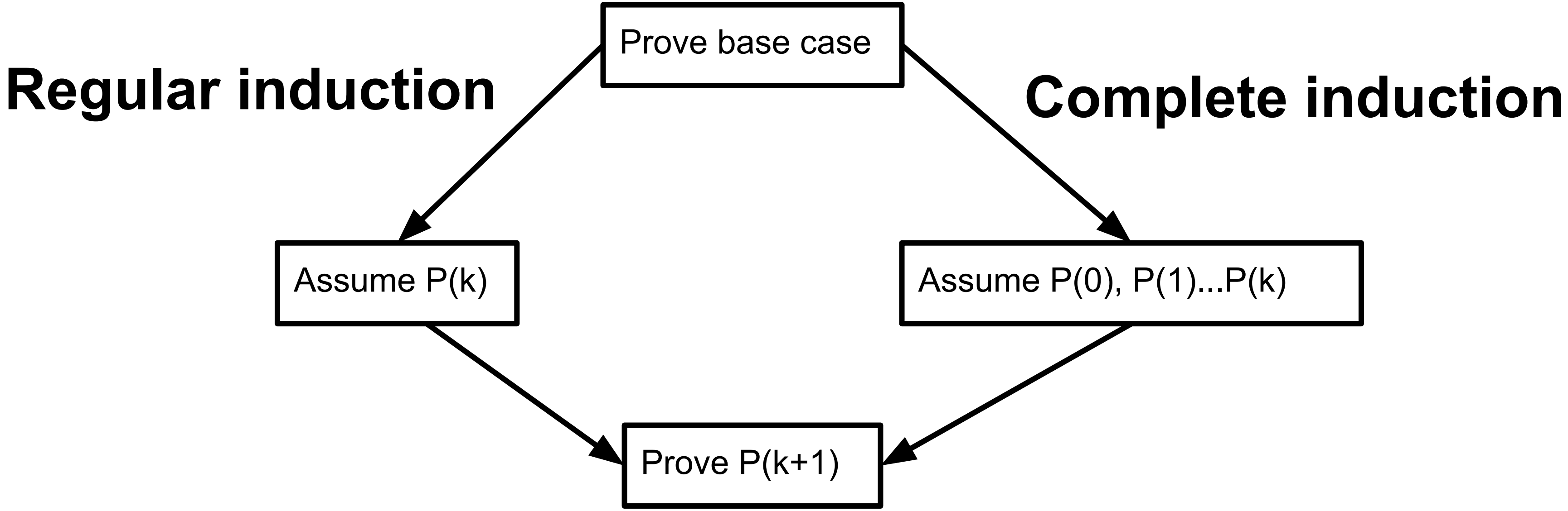
then

$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

Complete vs Regular Induction

- **Use regular induction if you can**
- Use complete if you need more than just P(k) when proving P(k+1)



Induction Clues + Tips

1) Look for **"all natural numbers n"** in the problem statement. Pretty much every induction problem uses that phrase (but there are non-induction problems that do, too!).

Let's begin with a refresher of the key terms and definitions involved. As a reminder, if L_1 and L_2 are languages over an alphabet Σ, then the ***concatenation of L_1 and L_2***, denoted L_1L_2, is the language

$$L_1L_2 = \{ wx \mid w \in L_1 \text{ and } x \in L_2 \}.$$

From concatenation, we can define ***language exponentiation*** of a language L inductively as follows:

$$L^0 = \{\varepsilon\} \qquad L^{n+1} = LL^n$$

You may find these formal terms helpful in the course of solving this problem.

Let A and B be arbitrary languages over some alphabet Σ. Prove, by induction, that if $X = AX \cup B$, then $A^nB \subseteq X$ for every $n \in \mathbb{N}$. Please use the formal definitions of concatenation, language exponentiation, union, and subset in the course of writing up your answer.

Induction Clues + Tips

2) Look for a link between smaller and larger problems (**recursion**!).

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Induction Clues + Tips

3) Think about **building up** for **existential P(n)** and **building down** for **universal P(n).**

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Induction Clues + Tips

4) Write down P(n) and make sure:

- P(n) will allow you to prove your end goal.
- The definition of P(n) includes n (<u>not all natural numbers n)</u>

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Example Induction Proof

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Example Induction Proof

Proof: Let A and B be arbitrary languages over some alphabet Σ where $X = AX \cup B$. Let $P(n)$ be the statement "$A^nB \subseteq X$." We will prove by induction that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

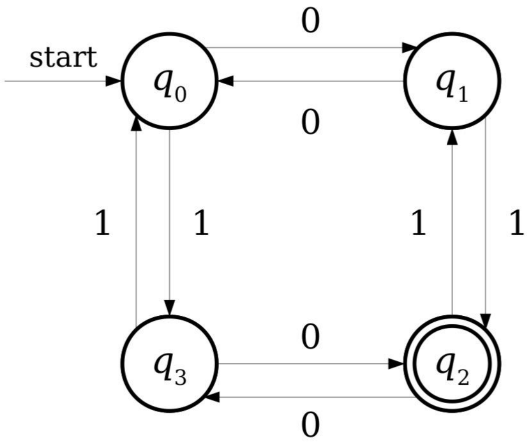
As our base case, we prove $P(0)$, that $A^0B \subseteq X$. Consider any $w \in A^0B$. This string must be of the form xy where $x \in A^0$ and $y \in B$. Since the only string in A^0 is ε, this means that $w = \varepsilon y = y$, so $w \in B$. Then, since $w \in B$, we know that $w \in AX \cup B$, and therefore that $w \in X$. Since our choice of w was arbitrary, this shows that every element of A^0B is an element of X, so $A^0B \subseteq X$, as required.

For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds and that $A^kB \subseteq X$. We will prove that $A^{k+1}B \subseteq X$. To do so, consider any arbitrary $w \in A^{k+1}B$. We will prove that $w \in X$.

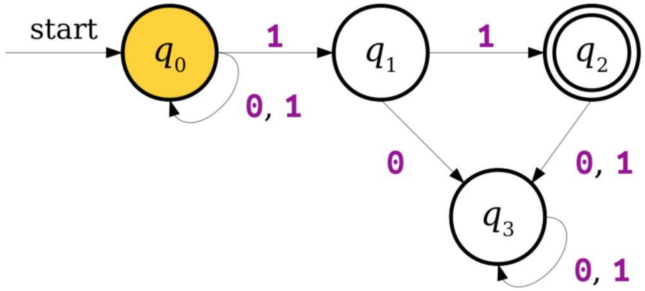
Since $A^{k+1}B = AA^kB = A(A^kB)$, we know see that $w \in A(A^kB)$. Consequently, there exist some $x \in A$ and $y \in A^kB$ such that $w = xy$. Since $y \in A^kB$, by our inductive hypothesis we see that $y \in X$. Overall, this shows that $w = xy$ where $x \in A$ and $y \in X$, so we see that $w \in AX$. Since $w \in AX$, we see that $w \in AX \cup B$, or equivalently that $w \in X$, as required. Thus $P(k+1)$ is true, completing the induction. ■

Regular Languages

Regular Languages



DFAs

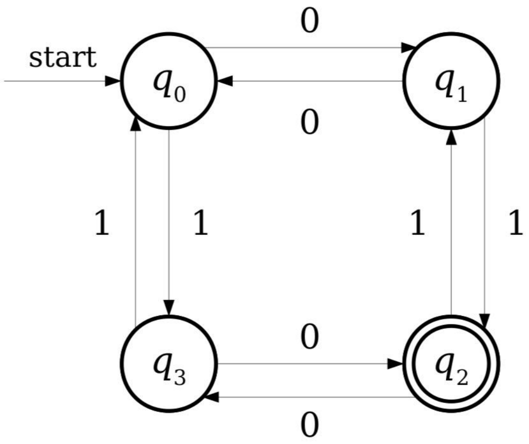


NFAs

`(a ∪ b)*aa(a ∪ b)*`

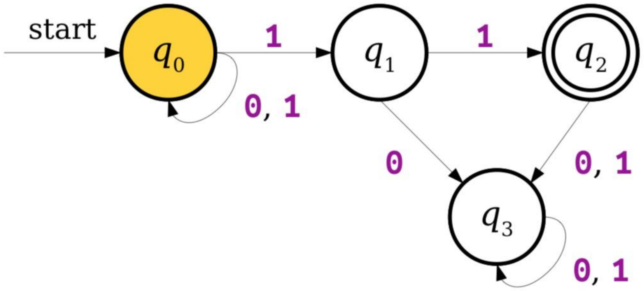
RegExs

Designing a DFA



states = pieces of information
transitions = when I read in a new character, how might this change what I know?

Designing an NFA



states = pieces of information
transitions = when I read in a new character, how might this change what I know?

AND

nondeterminism = assume you'll magically "know" when it's time to take the right transition

DFA Construction Example

Let $\Sigma = \{a, b\}$ and let $L_1 = \{\ w \in \Sigma^* \mid w$ does not contain bbb as a substring $\}$.

Design a DFA for L_1.

What are the states? How should my transitions link together the states?

DFA Construction Example

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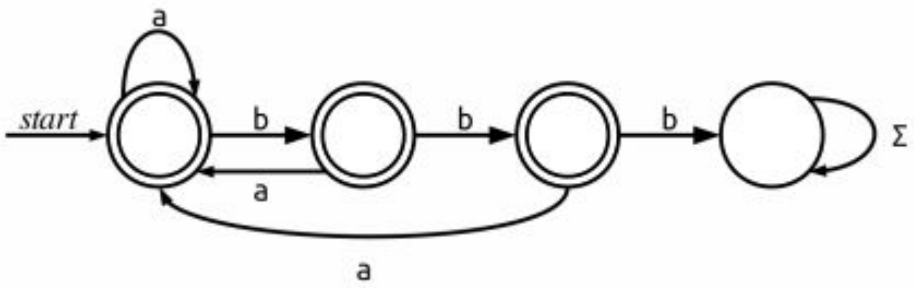
Design a DFA for L_1.

What are the states? How should my transitions link together the states?

Here is one possible solution:



This automaton works by advancing forward every time it sees a b and resetting whenever it sees an a. If it finds three consecutive b's, it enters a dead state.

Designing a RegEx

$$(a \cup b)^*aa(a \cup b)^*$$

1. Write out example strings and look for patterns
2. Can I separate the strings into different categories?
   a. If yes: UNION the categories together.
3. Can I break the strings into smaller subproblems?
   a. If yes: CONCATENATE each piece together.
4. Is there some sort of repeating structure?
   a. If yes: Use the KLEENE STAR on the smallest repeating pattern.

RegEx Construction Example

Let $\Sigma = \{a, b\}$ and $L = \{w \in \Sigma^* \mid w$ has an odd number of a's$\}$. Write a regular expression for L.

a

aaa

abb

bab

bba

aaa

aaabb

bbaaa

baaab

...

RegEx Construction Example

Let $\Sigma = \{a, b\}$ and $L = \{w \in \Sigma^* \mid w$ has an odd number of a's$\}$. Write a regular expression for L.

a
aaa
abb
bab
bba
aaa
aaabb
bbaaa
baaab
...

We need at least one a

We can have any number of b's, in any position

We can add a's two at a time to the original a

RegEx Construction Example

Let $\Sigma = \{a, b\}$ and $L = \{w \in \Sigma^* \mid w$ has an odd number of a's$\}$. Write a regular expression for L.

a
aaa
abb
bab
bba
aaa
aaabb
bbaaa
baaab
...

a

We need at least one a

RegEx Construction Example

Let Σ = {a, b} and L = {w $\in$ Σ* | w has an odd number of a's}. Write a regular expression for L.

a
aaa
abb
bab
bba
aaa
aaabb
bbaaa
baaab
...

Here is one possible solution:

b*ab*

We need at least one a

We can have any number of b's, in any position

RegEx Construction Example

Let Σ = {a, b} and L = {w ∈ Σ* | w has an odd number of a's}. Write a regular expression for L.

a
aaa
abb
bab
bba
aaa
aaabb
bbaaa
baaab
...

Here is one possible solution:

$$\mathbf{b^*ab^*(b^*ab^*ab^*)^*}$$

We can add a's two at a time to the original a

We need at least one a

We can have any number of b's, in any position

RegEx Construction Example

Let Σ = {a, b} and L = {w ∈ Σ* | w has an odd number of a's}. Write a regular expression for L.

a

aaa

abb

bab

bba

aaa

aaabb

bbaaa

baaab

...

Here is one possible solution:

b*ab*(b*ab*ab*)*

Myhill-Nerode Theorem

Myhill-Nerode Refresher

Theorem: Let L be a language over Σ.
If there is a set $S \subseteq \Sigma^*$ with the following properties, then L is not regular:

- S is infinite (that is, S contains infinitely many strings).
- The strings in S are ***pairwise distinguishable relative to L***. That is,

$$\forall x \in S.\ \forall y \in S.\ (x \neq y \rightarrow x \not\equiv_L y).$$

If you pick any two strings in S that aren't equal to one another…

… then they're distinguishable relative to L.

Distinguishability Refresher

- Let L be an arbitrary language over Σ.
- Two strings $x \in \Sigma^*$ and $y \in \Sigma^*$ are called ***distinguishable relative to L*** if there is a string $w \in \Sigma^*$ such that exactly one of xw and yw is in L.
- We denote this by writing $\boldsymbol{x \not\equiv_L y}$.

Myhill-Nerode Clues + Tips

1) Look for **"not a regular language"** in the problem statement. Pretty much every Myhill-Nerode problem involves proving that a language is not regular.

Let $\Sigma = \{a, b\}$. Consider the following language L_2 over Σ:

$$L_2 = \{\ a^n b^m \mid m, n \in \mathbb{N} \text{ and } m \leq 2n\ \}$$

For example, aa $\in L_2$ aab $\in L_2$ aabb $\in L_2$, aabbb $\in L_2$, and aabbbb $\in L_2$, but aabbbbb $\notin L_2$.

Prove that L_2 is not a regular language.

Myhill-Nerode Clues + Tips

2) Think about what you need to remember in order to prove that a string is in the language. Use that to pick an infinite set S. You need to prove that every pair of strings in S is distinguishable relative to L.

*The strings in S **do not** need to be in L**

Let $\Sigma = \{a, b\}$. Consider the following language L_2 over Σ:

$$L_2 = \{\ a^n b^m \mid m, n \in \mathbb{N} \text{ and } m \le 2n\ \}$$

For example, aa $\in L_2$ aab $\in L_2$ aabb $\in L_2$, aabbb $\in L_2$, and aabbbb $\in L_2$, but aabbbbb $\notin L_2$.

Prove that L_2 is not a regular language.

Example Myhill-Nerode Proof

Let $\Sigma = \{a, b\}$. Consider the following language L_2 over Σ:

$$L_2 = \{ a^n b^m \mid m, n \in \mathbb{N} \text{ and } m \leq 2n \}$$

For example, aa $\in L_2$ aab $\in L_2$ aabb $\in L_2$, aabbb $\in L_2$, and aabbbb $\in L_2$, but aabbbbb $\notin L_2$.

Prove that L_2 is not a regular language.

Proof: Let $S = \{ a^n \mid n \in \mathbb{N} \}$. The set S is infinite because it contains one string for each natural number. Now, consider any two strings $a^n, a^m \in S$. Without loss of generality, assume that $n < m$. Now, consider the strings $a^n b^{2m}$ and $a^m b^{2m}$. The string $a^n b^{2m}$ is not in L_2 because $2m > 2n$, so there are too many b's in the string for it to be in L_2. On the other hand, the string $a^m b^{2m}$ is in L because the number of b's is precisely twice the number of a's. Therefore, we see that $a^n \not\equiv_{L_2} a^m$. Since our choices of a^n and a^m were arbitrary, we therefore see that any two distinct strings in S are distinguishable relative to L_2. Therefore, since S is infinite, by the Myhill-Nerode theorem we see that L_2 is not regular. ■

Designing a CFG

$$S \to \varepsilon \mid a \mid b \mid aSa \mid bSb$$

1. Write out example strings and look for patterns
2. Think recursively - look for smaller strings within larger ones
3. "For every **x** I see, I need **y** somewhere else" means that **x, y** need to be added at the same time
4. Non-terminals represent different states/types of strings.

CFG Construction Example

Let $\Sigma = \{a, b\}$ and $L = \{w \in \Sigma^* \mid w$ has no a's or no b's$\}$.
Write a CFG for L.

bb

bbbbbb

bb

a

aaa

aa

...

CFG Construction Example

Let Σ = {a, b} and L = {w ∈ Σ* | w has no a's or no b's}.
Write a CFG for L.

bb
bbbbbb
bb
a
aaa
aa
...

We can either have any number of a's or any number of b's

CFG Construction Example

Let Σ = {a, b} and L = {w ∈ Σ* | w has either no a's or no b's}. Write a CFG for L.

We start by "choosing" a's or b's

After we've chosen, we can add as many of the chosen letter as we want

Start symbol: **S**

S → ε | a**A** | b**B**

A → a**A** | ε

B → b**B** | ε

Lava Diagram

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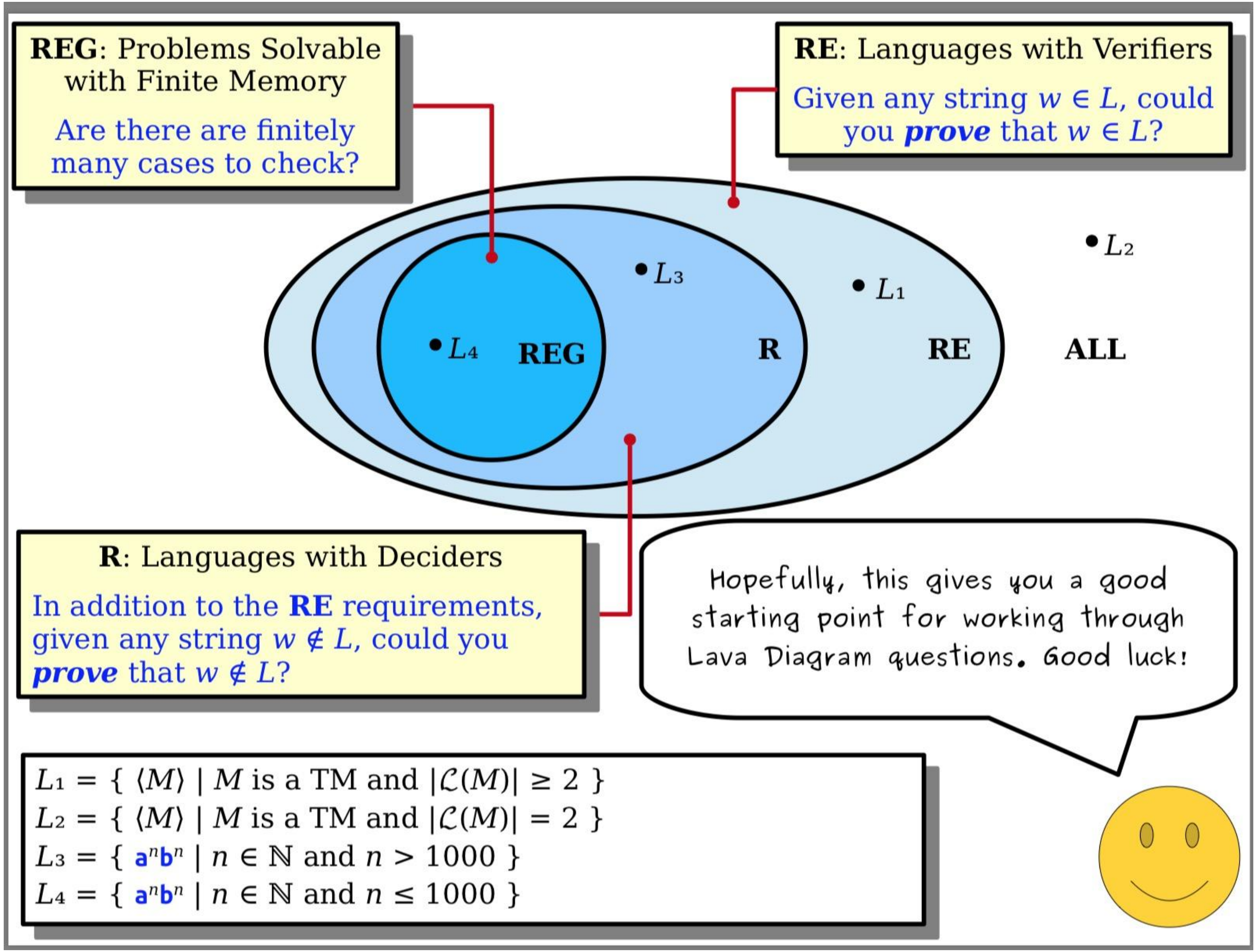
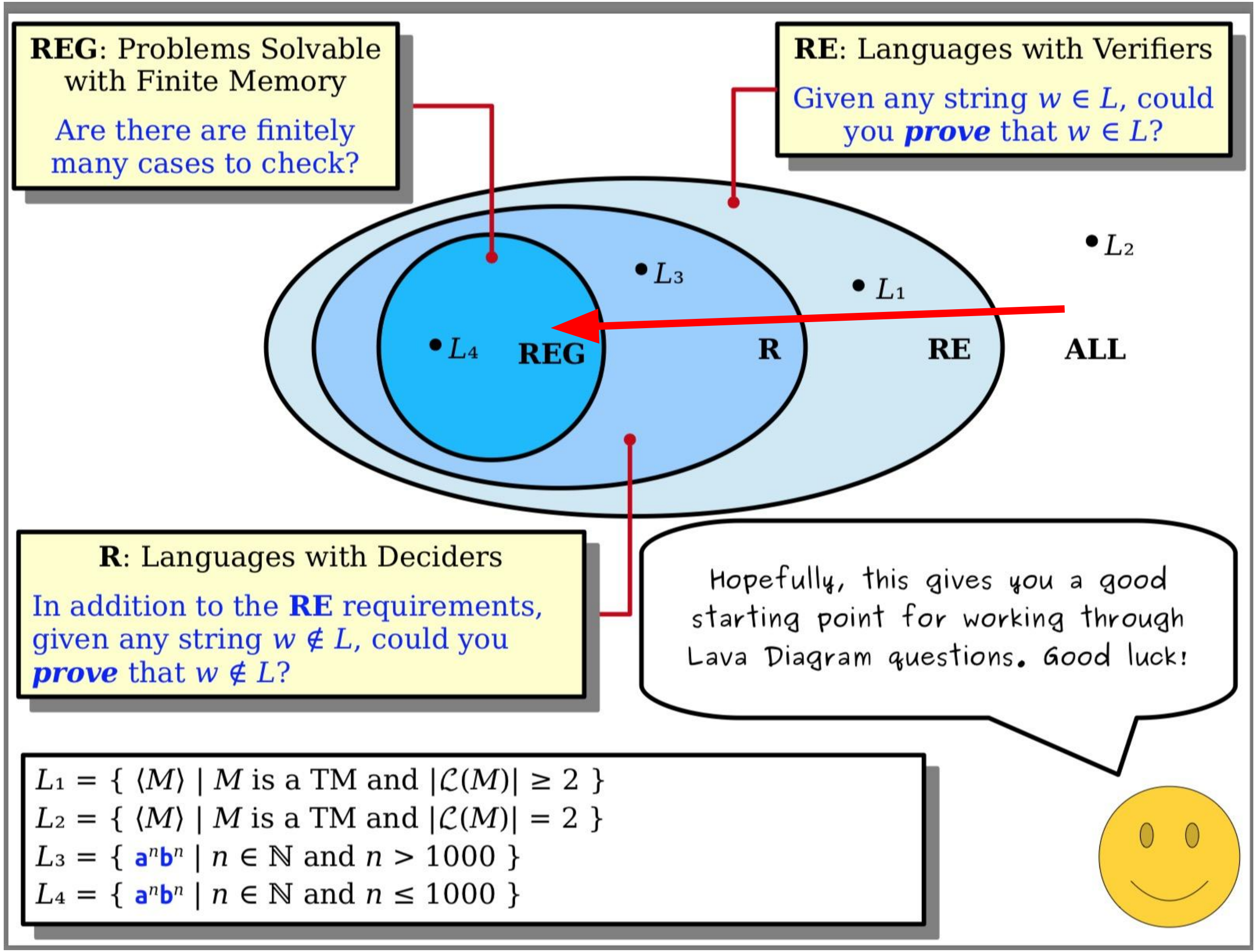


Lava Diagram

Intuition:

Start in ALL and see if you can move the language in more



Questions?